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Quantitative characterization of the difference between Birkhoff orthogonality and isosceles orthogonality ☆

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Abstract

In this paper we introduce a new geometry constant $D(X)$ to give a quantitative characterization of the difference between Birkhoff orthogonality and isosceles orthogonality. We show that 1 and $2(\sqrt{2}-1)$ is the upper and lower bound for $D(X)$, respectively, and characterize the spaces of which $D(X)$ attains the upper and lower bounds. We calculate $D(X)$ when $X=(R^2,\|\cdot\|_p)$ and when X is a symmetric Minkowski plane respectively, we show that when X is a symmetric Minkowski plane $D(X)=D(X^*)$.

Keywords: Birkhoff orthogonality; Isosceles orthogonality

1. Introduction

One of the ideas that plays a fundamental role in Euclidean geometry is that of orthogonality. Not only does this concept occur in Euclid's axioms themselves, but also in many of the basic theorems. One of the underlying themes in Banach space theory is to look for suitable substitutes for this notion. Roberts [6] introduced Roberts orthogonality in 1934: let X be a real normed linear space, for any $x, y \in X$, x is said to be Roberts orthogonal to $y (x \perp_R y)$ iff

$$\|x+ty\| = \|x-ty\|, \quad \forall t \in \mathbb{R}.$$

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Birkhoff [3] introduced Birkhoff orthogonality: x is said to be Birkhoff orthogonal to y ($x \perp_B y$) iff

$$\|x + ty\| \geqslant \|x\|, \quad \forall t \in \mathbb{R}.$$

James [4] introduced isosceles orthogonality which has been shown by Ji Donghai [7] closely connected with the nonsquareness of normed linear spaces: x is said to be isosceles orthogonal to y ($x \perp_I y$) iff

$$\|x + y\| = \|x - y\|.$$

In this paper we mainly consider the last two notions of orthogonality. Generally Birkhoff orthogonality and isosceles orthogonality are different and they coincide iff the underlying space is an inner product space [2]. Intuitively the different between those two types of orthogonality in different spaces are different, to answer how difference those two orthogonalities are, we introduced the constant $D(X)$ for a real normed linear space X as:

$$D(X) = \inf\Big\{\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\|\colon\ x, y \in S(X), x \perp_I y\Big\},$$

where $S(X)$ is the unit sphere of X.

2. The lower and upper bounds of $D(X)$

Theorem 1. *For any real normed linear space X with* $\dim(X) \geqslant 2$*,*

$$2(\sqrt{2} - 1) \leqslant D(X) \leqslant 1$$

and $D(X) = 1$ iff X is Euclidean.

Proof. It is apparently that for all $x, y \in S(X)$ with $x \perp_I y$, we always have $\inf_{\lambda \in \mathbb{R}}\{\|x + \lambda y\|\} \leqslant 1$. Then for all $x, y \in S(X)$ with $x \perp_I y$, we always have $\inf_{\lambda \in \mathbb{R}}\{\|x + \lambda y\|\} = 1$ under the condition that $D(X) = 1$, that is, isosceles orthogonality implies Birkhoff orthogonality on $S(X)$, by [2], X is then an inner product space. Because those two orthogonalities coincide in inner product spaces the converse is also true.

On the other hand, for James has already proved that $\forall x, y \in X$ with $x \perp_I y$ and $\|y\| \leqslant \|x\|$,

$$\|x + ky\| \geqslant 2(\sqrt{2} - 1)\|x\|, \quad \forall k \in \mathbb{R},$$

and if the equality holds then $|k| = \sqrt{2} - 1$ in [4], we will always have $D(X) \geqslant 2(\sqrt{2} - 1)$. □

Theorem 2. *For any real Banach space X with* $\dim(X) \geqslant 2$*, there exist $e_1, e_2 \in S(X)$ such that $e_1 \perp_I e_2$ and*

$$\inf_{\lambda \in R}\big\{\|e_1 + \lambda e_2\|\big\} = 2(\sqrt{2} - 1)$$

iff there exists a two-dimensional subspace X_0 of X and $x_0 \in S(X_0)$ such that x_0 is the common endpoints of two segments of which the length are not less than $\sqrt{2}$.

Proof. Suppose that there exist $e_1, e_2 \in S(X)$ such that $e_1 \perp_I e_2$ and

$$\inf_{\lambda \in R}\big\{\|e_1 + \lambda e_2\|\big\} = 2(\sqrt{2} - 1).$$

Then $\exists X_0 \subset X$ with $\dim(X_0) = 2$, $e_1, e_2 \in S(X_0)$ and

$$\|e_2 + (\sqrt{2}-1)e_1\| = 2(\sqrt{2}-1).$$

Let $x = \frac{1}{2}e_1 + \frac{\sqrt{2}+1}{2}e_2$, $u_1 = \frac{\sqrt{2}}{2}(e_1+e_2)$, $u_2 = \frac{\sqrt{2}}{2}(e_2 - e_1)$ then

$$\|x\| = \frac{1}{2}\|e_1 + (\sqrt{2}+1)e_2\| = \frac{1}{2(\sqrt{2}-1)}\|e_2 + (\sqrt{2}-1)e_1\| = 1,$$
$$\|u_1\| = \left\|\frac{\sqrt{2}}{2}(e_1+e_2)\right\| = \|(2-\sqrt{2})x + (\sqrt{2}-1)e_1\| \leqslant 1,$$
$$\|u_2\| = \left\|\frac{\sqrt{2}}{2}(e_2-e_1)\right\| = \|-\sqrt{2}x + (1+\sqrt{2})e_2\| \geqslant \|(1+\sqrt{2})e_2\| - \|\sqrt{2}x\| = 1.$$

Then $1 \leqslant \|u_2\| = \|u_1\| \leqslant 1$, so $\|u_1\| = \|u_2\| = 1$.

At the same time we can easily get the following two equalities:

$$u_1 = (2-\sqrt{2})x + (\sqrt{2}-1)e_1,$$
$$e_2 = (\sqrt{2}-1)u_2 + (2-\sqrt{2})x$$

that is u_1 is on the segment $[e_1, x]$; e_2 is on the segment $[u_2, x]$. We have already proved

$$\|e_i\| = \|x\| = \|u_j\| = 1, \quad i, j = 1, 2,$$

so $[e_1, x] \subset S(X_0)$, $[u_2, x] \subset S(X_0)$. We now just have to show that the length of those two segments are both $\sqrt{2}$.

In fact,

$$\|x - e_1\| = \frac{1}{2(\sqrt{2}-1)}\|e_2 - (\sqrt{2}-1)e_1\| = \sqrt{2}\left\|\frac{2-\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}u_2\right\| = \sqrt{2},$$
$$\|x - u_2\| = \|(1+\sqrt{2})x - (\sqrt{2}+1)e_2\| = \frac{1}{2}\|(\sqrt{2}+1)e_1 + e_2\|$$
$$= \sqrt{2}\left\|\left(1 - \frac{\sqrt{2}}{2}\right)x + \frac{\sqrt{2}}{2}e_1\right\| = \sqrt{2}.$$

The necessity has been proved.

Now suppose that there exists a two-dimensional subspace X_0 of X, $w, x, z \in S(X_0)$ such that $[w, x] \subset S(X_0)$, $[z, x] \subset S(X_0)$ and $\|w - x\| = \|z - x\| = \sqrt{2}$. Then $\exists e_1^0 \in [w, x]$, $e_2^0 \in [z, x]$ such that $\|e_1^0 - x\| = \|e_2^0 - x\| = 1$ and $x = e_1^0 + e_2^0$. For $\|w - x\| = \sqrt{2}$ and $\|e_1^0 - x\| = 1$, we can easily show that

$$w = \sqrt{2}e_1^0 + (1-\sqrt{2})x = e_1^0 + (1-\sqrt{2})e_2^0.$$

Similarly, $z = (1-\sqrt{2})e_1^0 + e_2^0$.

Let $e_1 = w$, $e_2 = e_2^0 + (\sqrt{2}-1)e_1^0$, then $\|e_1\| = \|w\| = 1$ and

$$\|e_2\| = \|e_2^0 + (\sqrt{2}-1)e_1^0\| = \|(\sqrt{2}-1)x + (2-\sqrt{2})e_2^0\| = 1.$$

We will now show that e_1, e_2 are two elements satisfy the condition.

In fact,

$$\begin{aligned}\|e_2+(\sqrt{2}-1)e_1\| &= \|e_2^0+(\sqrt{2}-1)e_1^0+(\sqrt{2}-1)e_1^0-(\sqrt{2}-1)^2e_2^0\|\\ &= 2(\sqrt{2}-1)\|e_1^0+e_2^0\| = 2(\sqrt{2}-1)\|x\| = 2(\sqrt{2}-1),\\ \|e_1+e_2\| &= \|\sqrt{2}e_1^0+(2-\sqrt{2})e_2^0\| = \sqrt{2}\|e_1^0+(\sqrt{2}-1)e_2^0\|\\ &= \sqrt{2}\|(\sqrt{2}-1)x+(2-\sqrt{2})e_1^0\| = \sqrt{2}.\end{aligned}$$

Similarly, $\|e_1-e_2\|=\sqrt{2}$. This completes the proof. □

3. The constant in l_p^2 spaces

Theorem 3.

$$D(l_p^2)=\inf\left\{\frac{(1+t^2)}{(1+t^q)^{1/q}(1+t^p)^{1/p}}:\ t\in[0,1]\right\}.$$

Proof. A given point (α,β) is on the unit circle of l_p^2 iff

$$|\alpha|^p+|\beta|^p=1.$$

Without lose of generality we can suppose that $\alpha,\beta\geqslant 0$. Let $x=(\alpha,\beta)$, $y=(-\beta,\alpha)$, then

$$x+\lambda y=(\alpha-\beta\lambda,\beta+\alpha\lambda).$$

Let

$$f(\lambda)=\|x+\lambda y\|^p=|\alpha-\beta\lambda|^p+|\beta+\alpha\lambda|^p$$

and

$$f_1(\lambda)=(\alpha-\beta\lambda)^p+(\beta+\alpha\lambda)^p,$$

then $f_1(\lambda)$ coincides with $f(\lambda)$ on $[-\frac{\beta}{\alpha},\frac{\alpha}{\beta}]$. Since $f(\lambda)$ is a convex function of λ, $f_1(\lambda)$ is also a convex function on $[-\frac{\beta}{\alpha},\frac{\alpha}{\beta}]$. The derivative of $f_1(\lambda)$ is

$$f_1'(\lambda)=p\bigl(-\beta(\alpha-\beta\lambda)^{p-1}+\alpha(\beta+\alpha\lambda)^{p-1}\bigr).$$

By letting $f_1'(\lambda)=0$, we have

$$\lambda=\frac{\alpha-\beta\gamma}{\alpha\gamma+\beta},$$

where $\gamma=(\frac{\alpha}{\beta})^{1/(p-1)}$. Obviously $\lambda\leqslant\frac{\alpha}{\beta}$ and by the inequality

$$\lambda-\left(-\frac{\beta}{\alpha}\right)=\frac{\alpha^2+\beta^2}{\alpha(\alpha\gamma+\beta)}\geqslant 0,$$

we have $\lambda\in[-\frac{\beta}{\alpha},\frac{\alpha}{\beta}]$. Thus $f_1(\lambda)$ attains its minimum on $[-\frac{\beta}{\alpha},\frac{\alpha}{\beta}]$, and then $f(\lambda)$ attains its minimum on $[-\frac{\beta}{\alpha},\frac{\alpha}{\beta}]$.

Let $t=\frac{\alpha}{\beta}$ and $\lambda_0=\frac{t-t^{q-1}}{1+t^q}$, where q is the real number such that $\frac{1}{p}+\frac{1}{q}=1$ holds. Then we have $f_1'(\lambda_0)=0$ and

$$
\begin{aligned}
f_1(\lambda_0) &= (\alpha-\beta\lambda_0)^p + (\beta+\alpha\lambda_0)^p = \beta^p\left(\left(\frac{\alpha}{\beta}-\lambda_0\right)^p + \left(1+\frac{\alpha}{\beta}\lambda_0\right)^p\right) \\
&= \beta^p\big((t-\lambda_0)^p + (1+t\lambda_0)^p\big) = \beta^p\left(\left(t-\frac{t-t^{q-1}}{1+t^q}\right)^p + \left(1+t\frac{t-t^{q-1}}{1+t^q}\right)^p\right) \\
&= \beta^p\left(\left(\frac{t^{q+1}+t^{q-1}}{1+t^q}\right)^p + \left(\frac{1+t^2}{1+t^q}\right)^p\right) \\
&= \beta^p\left(t^{(q-1)p}\left(\frac{1+t^2}{1+t^q}\right)^p + \left(\frac{1+t^2}{1+t^q}\right)^p\right) = \beta^p\big(t^q+1\big)\left(\frac{1+t^2}{1+t^q}\right)^p \\
&= \left(\frac{1+t^q}{1+t^p}\right)\left(\frac{1+t^2}{1+t^q}\right)^p = \frac{(1+t^2)^p}{(1+t^q)^{p-1}(1+t^p)}.
\end{aligned}
$$

Thus

$$
D\big(l_p^2\big) = \inf\left\{\frac{(1+t^2)}{(1+t^q)^{1/q}(1+t^p)^{1/p}} \colon t \in [0,1]\right\}. \qquad \square
$$

Corollary 4. *$D(l_p^2) = D(l_q^2)$, where q is the real number such that $\frac{1}{p}+\frac{1}{q}=1$.*

Corollary 5. $\lim_{p\to\infty} D(l_p) = 2(\sqrt{2}-1)$.

4. $D(X)$ in symmetric Minkowski planes

Let X be a Minkowski plane (= real two-dimensional normed linear space), $S(X)$ be the unit circle of X, if there exist $e_1, e_2 \in S(X)$ such that:

$$
\|e_1 + te_2\| = \|e_1 - te_2\| = \|e_2 + te_1\| = \|e_2 - te_1\|
$$

holds for all $t \in \mathbb{R}$, then we call X a symmetric Minkowski plane and $\{e_1, e_2\}$ a pair of axes of X.

Example 6. Let $X = (R^2, \|\cdot\|_p)$ $(1 \leqslant p \leqslant \infty)$ then X is a symmetric Minkowski plane and $\{(1,0),(0,1)\}$ is a pair of axes of X.

Example 7. Let $X = (R^2, \|\cdot\|_8)$ where $\|\cdot\|_8 = \max\{\|\cdot\|_\infty, \frac{1}{\sqrt{2}}\|\cdot\|_1\}$, then $S(X)$ is a regular octagon and X is a symmetric Minkowski plane.

Example 8. Let $X = (R^2, \|\cdot\|_{p_1+p_2})$ $(1 \leqslant p_1 \leqslant p_2 \leqslant \infty)$ where $\|\cdot\|_{p_1+p_2} = \frac{\|\cdot\|_{p_1}+\|\cdot\|_{p_2}}{2}$ then X is a symmetric Minkowski plane and $\{(1,0),(0,1)\}$ is a pair of axes of X.

In a sense symmetric Minkowski planes are the generalization of l_p-planes and regular $4n$-gonal normed planes, and they have the following basic properties:

Theorem 9. *Let X be a symmetric Minkowski plane and $\{e_1, e_2\}$ be a pair of axes of X then*:

(1) $e_1 \perp_R e_2$,
(2) $e_1 \perp_I e_2$,
(3) *e_1 and e_2 is a pair of conjugate diameters of X,*

(4) *X^* is also a symmetric Minkowski plane and $\{e_1^*, e_2^*\}$ is a pair of axes of X^*, where e_1^*, e_2^* is the supporting functional of e_1, e_2, respectively.*

Proof. (1), (2) is trivial. To prove (3) we just need to notice that the function $f(t) = \|e_1 + te_2\|$ is a even and convex function of t, so $f(t)$ must attain its minimal value at 0, that is $e_1 \perp_B e_2$. Similarly we can prove that $e_2 \perp_B e_1$, then e_1 and e_2 is a pair of conjugate diameters of X.

Now we just have to show (4). In fact:

$$\left\|e_1^* + te_2^*\right\| = \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|(e_1^* + te_2^*)(\alpha e_1 + \beta e_2)|}{\|\alpha e_1 + \beta e_2\|} = \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|\alpha + t\beta|}{\|\alpha e_1 + \beta e_2\|}.$$

Similarly,

$$\left\|e_1^* - te_2^*\right\| = \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|(e_1^* - te_2^*)(\alpha e_1 + \beta e_2)|}{\|\alpha e_1 + \beta e_2\|} = \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|\alpha - t\beta|}{\|\alpha e_1 + \beta e_2\|}.$$

Then $\|e_1^* + te_2^*\| = \|e_1^* - te_2^*\|$ and $\|e_2^* + te_1^*\| = \|e_2^* - te_1^*\|$ in a similarly way. Now we just have to show that $\|e_1^* + te_2^*\| = \|e_2^* + te_1^*\|$

$$\begin{aligned}\left\|e_2^* + te_1^*\right\| &= \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|(e_2^* + te_1^*)(\alpha e_1 + \beta e_2)|}{\|\alpha e_1 + \beta e_2\|} = \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|\beta + t\alpha|}{\|\alpha e_1 + \beta e_2\|} \\ &= \sup_{x=\alpha e_1+\beta e_2\in X} \frac{|\beta + t\alpha|}{\|\beta e_1 + \alpha e_2\|} = \left\|e_1^* + te_2^*\right\|.\end{aligned}$$

This completes the proof. □

Theorem 10. *Let X be a symmetric Minkowski plane, $\{e_1, e_2\}$ be a pair of axes of X, then $\forall x, y \in S(X), x = \alpha e_1 + \beta e_2$, $x \perp_I y$ iff $y = \pm(-\beta e_1 + \alpha e_2)$.*

Proof. By the uniqueness of isosceles property on the unit circle [1, Corollary 4] we just need to show $y = -\beta e_1 + \alpha e_2$ is one of two elements on $S(X)$ such that $x \perp_I y$ and this follows immediately from equalities below:

$$\begin{aligned}\|x + y\| &= \|\alpha e_1 + \beta e_2 + \alpha e_2 - \beta e_1\| = \left\|(\alpha - \beta)e_1 + (\alpha + \beta)e_2\right\| \\ &= \left\|(\alpha + \beta)e_1 + (\alpha - \beta)e_2\right\| = \left\|(\alpha + \beta)e_1 + (\beta - \alpha)e_2\right\| \\ &= \left\|\alpha e_1 + \beta e_2 - (-\beta e_1 + \alpha e_2)\right\| = \|x - y\|. \qquad \square\end{aligned}$$

By the above two theorems we can calculate $D(X)$ for the symmetric Minkowski plane X, and it can be seen that the following theorem is a generalized version of Theorem 3.

Theorem 11. *Let X be a symmetric Minkowski plane on R^2 and $e_1 = (1, 0)$, $e_2 = (0, 1)$ be a pair of axes of X, then*

$$D(X) = \inf\left\{\frac{1 + t^2}{\|(t, 1)\|\,\|(t, 1)\|_*}\colon t \in \mathbb{R}\right\},$$

where $\|\cdot\|_*$ denotes the norm of the dual space of X.

Proof. Without lose of generality we can suppose that X is smooth and strictly convex. For any $x=(\alpha,\beta)\in S(X)$, we have $y=(-\beta,\alpha)\in S(X)$ and $x\perp_I y$.

On the other hand, if $\exists\lambda_0\in\mathbb{R}$ such that $\|x+\lambda_0 y\|=\inf_{\lambda\in\mathbb{R}}\|x+\lambda y\|$ then we have

$$\|x+\lambda_0 y+\lambda y\|\geqslant\|x+\lambda_0 y\|,\quad \forall\lambda\in\mathbb{R}.$$

that is $x+\lambda_0 y\perp_B y$. By [5] there exists $f\in S(X^*)$ such that $f(y)=0$ and

$$f(x+\lambda_0 y)=f(x)=\|x+\lambda_0 y\|. \tag{1}$$

One can easily verify that $f=\frac{(\alpha,\beta)}{\|(\alpha,\beta)\|_*}$ so (1) turns out to be

$$\frac{(\alpha,\beta)}{\|(\alpha,\beta)\|_*}(\alpha,\beta)=\|x+\lambda_0 y\| \tag{2}$$

that is

$$\frac{\alpha^2+\beta^2}{\|(\alpha,\beta)\|_*}=\|x+\lambda_0 y\|. \tag{3}$$

With out lose of generality we can suppose that $\alpha,\beta\neq 0$ then (3) is equivalent to

$$\frac{1+t^2}{\|(t,1)\|\|(t,1)\|_*}=\|x+\lambda_0 y\| \tag{4}$$

immediately, we can have

$$D(X)=\inf\left\{\frac{1+t^2}{\|(t,1)\|\|(t,1)\|_*}:\ t\in\mathbb{R}\right\}.\qquad\square$$

Corollary 12. *If X is a symmetric Minkowski plane then $D(X)=D(X^*)$.*

Remark 13. We conjecture that there exist spaces that there is no isosceles orthogonal elements $x,y\in S(X)$ such that $\inf_{\lambda\in\mathbb{R}}\{\|x+\lambda y\|\}=D(X)$.

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